3. It gives a vital and essential but not a costly concrete correlation of studies. This is a correlation of and through functions and utilities.

4. It is readily seen that all functions vary with circumstances, hence this treatment tears apart the narrow rigid processes of the demonstrational treatment, and leaves the pupil free and stimulates him to variations and innovations. This, rather than the demonstrational one, seems to be the route along which some of the greatest discoveries in mathematics have been made. Hence our method gives new value to the study of the history of a subject.

5. Finally the method under consideration helps to define the cultural values of education, and put these values in a more definite and controllable shape. This is a help which is very much needed at this time.

LAWRENCEVILLE SCHOOL,
LAWRENCEVILLE, N. J.

MODERN TENDENCIES IN THE TEACHING OF ALGEBRA.*

BY N. J. LENNES.

One of the most obvious facts about mathematics in our secondary schools is a very general dissatisfaction which is expressed on all sides. There is an alarming number of failures, especially in the first year of the high school, which argues that the pupils do not find the subject suited to their tastes and capacities. Instructors in the colleges and universities rarely miss an opportunity for declaring that their students came poorly prepared. The programs of teachers' meetings and the tables of contents of pedagogical journals are teeming with titles which assume that something is wrong.

Is this general faultfinding simply the natural condition of normal progress, the sign of healthy life, or is there at present an abnormal divergence between our practices and our best principles? and if the latter, what are the causes of this divergence? The main facts of the general history of pedagogy

* Read at the meeting of the New York Section.

during the last few decades and of the teaching of secondary mathematics during this time furnish the answers to these questions.

One of the chief characteristics of modern pedagogy is the importance which it places upon the study of the child itself,— its needs, interests and capacities. It has become a commonplace that in growing from childhood to maturity the individual passes through a series of stages, and that in each stage the intellectual as well as the physical powers are quite different from those of the stages that precede and follow. The problem that the modern educator has set himself is to make such demands upon the pupil as will result in the most complete and most spontaneous self-expression at each stage.

The spontaneity of this self-expression has come to be of the greatest moment. In days not long gone by, the course of study was prepared without much question as to whether the work demanded was of natural interest to the pupil. By stern command he was bid to do his task. The more distasteful the work the greater its disciplinary value. All this has been radically changed. We now know that a person, young or old, may be compelled to perform a certain amount of labor, physical or mental, but that so long as his interest is not awakened so long does he fall far short of doing the best work of which he is capable. President Eliot is often quoted as saying that any man who is worth his salt, whether he works with his brains alone or with his brains and hands, takes delight first in the work itself, second, in what he produces, and third, in what it will bring him. There is a stupendous difference in capacity and also in happiness between the man who works only at the bidding of another and the man whose work has taken possession of him.

In this respect all people, young and old, are alike. I still remember my introduction to quadratic equations. The teacher gave us the dimensions of an ordinary school slate and the area of its frame and required us to find the width of the frame. The next day we had all failed to solve this problem. We were convinced that our equipment in algebraic processes were insufficient and were quite put out that such a simple problem should be beyond us. Learning to solve a quadratic equation was a mere detail after this interest had been created. What would

have been the comparative effect if the teacher had given us a quadratic equation and asked us to solve it, possibly seeking to interest us by saying that we should wish to obtain a complete knowledge of algebra?

But if it be admitted that it is necessary to lay hold on the natural interests of the child in order to secure the best results then the question: What manner of creature is this child? becomes a momentous one. It is in attempting to answer this question that the modern investigators in pedagogy have been led to make such profound study of child life. The results of this study so far as they effect the questions here at issue are unequivocal and accord entirely with common sense.

What are the characteristics of the child of fourteen for whom we are to build our course in algebra? He is a bundle of activities, not a philosopher. He is interested in phenomena rather than in their explanation. Most of all is he interested in that which enables him to do things,—to express himself in activity. Most remote from his interests are theoretical considerations which to him are not connected with the teeming world in which he lives and which at every moment is flooding his receptive and curious mind with sensations.

The understanding of these simple facts has created a pedagogical awakening which has wrought fundamental changes in the teaching of nearly all subjects in the secondary schools. A few decades ago the beginner's Latin grammar was a complete treatise of that great language. The same text was used by the child studying his first lesson in Latin and the graduate student in the universities. Now the beginner's Latin book is decidedly juvenile to the mature student. There are frequent comparisons with the mother tongue, and even more constantly than is apparent from the texts does the good teacher maintain interest by making connection with facts of language which the child already knows. Thus the study of Latin is made less logical, but it is vastly more interesting, and it is also more profound in that it takes cognizance of the interrelations of languages and of general principles of language as distinct from the principles of one particular language.

The teaching of other subjects has undergone similar changes. English was a study of grammatical forms, of styles of speech and writing. A vast amount of time was spent in analyzing

sentences, classifying figures of speech and in memorizing the accepted canons of rhetoric. Here also the aim was to obtain a logically complete view. But little emphasis was laid on actual writing.

History was a record of the births and deaths of kings and queens—a chronicle of wars and of great battles. Now we attempt to memorize fewer dates, but seek to understand more clearly the forces that are at work. We try to comprehend the acts of men in distant times by observing men as they act now and by trying to place ourselves and others in the positions of those we are studying.

Physics consisted in the statement of a few fundamental facts and in the deduction from these of other facts. An accumulation of facts was the prime object. Now the laboratory has so changed the study of physics that the pupil of fifty years ago would not recognize it under the old name.

In all the subjects there has been a sacrifice in logical formality and in the number of facts that are lodged in the memory. At the same time there has been a vast gain in real insight. The subjects have been clothed with a human interest which was foreign to them and which is doing wonders for the spread of general intelligence. That the transition has brought some losses no one denies, but on the whole it is agreed that the net gain has been great. There is not the slightest sign of a movement to return to the old order of things. By far the greater part of the changes here enumerated have been brought about during the last twenty years. The description just given of the old order of things is largely taken from my own experience as a student in American high schools.

It is a remarkable fact that throughout this period of adjustment to a fundamentally new point of view the teaching of secondary mathematics has remained practically unchanged. This is particularly true of algebra, with which we are concerned this evening. To verify this statement one has but to compare the texts of today with those of twenty or even thirty years ago. There have been improvements in typography, in simplicity and perhaps in accuracy of statement and in the number and character of the exercises. There have been additions of new matter such as graphs and problems from physics. These are usually put into the books in the form of appendages, either as formal

appendices or tacked on at the end of chapters instead of being woven in as organic parts of the books. There have also been additions of new cases of factoring, new cases in simultaneous quadratics, etc. There is an evident effort to treat every possible case. Indeed one of the standard algebras of today reminds one of the Latin grammars or the rhetorics of twenty years ago.

The essential character of these books has remained unchanged. The topics considered, the order of their introduction and the general mode of treatment are precisely what they were a quarter of a century ago. Thus it comes about that our courses in algebra have descended to us from a time when the accepted principles of pedagogy were quite different from those we hold to-day. It is difficult to assign reasons why the teaching of secondary mathematics has adjusted itself more slowly to the requirements of modern pedagogy than that of any other great subject, but there can be little doubt that this exceptionally tardy conformity of practice to our best principles is the reason for the prevailing discontent noted at the beginning.

But not only has there been a general failure to keep step with the growing pedagogical principles; there has even been a movement in the opposite direction. The desire for completeness has ever been at work introducing new cases of abstract manipulation. This has affected particularly the first part of the course. The rivalry of authors to make their books " strong on factoring " has extended this subject beyond all reasonable limits for a beginner's course. Complicated fractions and equations containing fractions, long problems in multiplication and division have filled the early part of algebra with operations on abstract symbols so that now the first half of the first year's work in algebra contains practically no concrete problems. There are usually a few such problems at the very beginning, but these are so simple that they are solved before the study of algebra begins. From twelve to fifteen weeks, and often more, the child is made to study abstract theory and work formal exercises.

That this is not in conformity with our best principles of pedagogy is now pretty generally recognized. In no other subject is the pupil kept in such complete isolation from the rest of the world for so long a time. In beginning the study of languages, ancient or modern, our teachers

are constantly making connection with the mother tongue. In history and the study of government we are all the while making our departure from present conditions with which the child is acquainted. In the sciences the laboratory is as much a necessity as the teacher. Even in advanced courses in mathematics, given in the graduate schools of our great universities, the lecturer frequently shows the connection between the subject he is teaching and other subjects with which the students are acquainted independently of that particular course. I well remember attending a course of lectures on quaternions in which after a few days of purely theoretical work the professor showed how the theory he had developed could be used to make very elegant proofs of certain theorems in spherical trigonometry. The effect on this class of mature and comparatively expert mathematicians was unmistakable. The eyes shone brighter, the pens worked more rapidly. Suddenly their interest had been awakened. I believe it quite safe to say that nearly all of our well-known lecturers on higher mathematics seek constantly to make connection with the student's general information. Suppose that we were to attend a course in which the lecturer should begin by laying down his axioms and definitions and that then for fifteen weeks, four or five hours a week, he were to go on developing a symbolism peculiar to that course, proving theorems and working exercises in terms of that symbolism. Never once would he give us a problem which we could understand as a problem aside from the particular subject we were studying,—never once make connection with other subjects we know. I am quite certain that even with our mature years we should find it a great test of our endurance. Suffice it to say that the lecturer will not try the experiment. For one thing he is too good a teacher and for another he knows too well that his classes would soon dwindle to nothing. The people attending them have arrived at years of discretion. The children who attend our classes have nothing to say about such matters.

> "Theirs not to reason why,
> Theirs but to do and die,
> Into the valley of death
> Go the poor youngsters."

If it is admitted, and it is scarcely open to doubt, that the

child is interested rather in events than in their explanation,—that his interest is aroused by the dynamic rather than the philosophical aspect of things, then it follows that, in the beginning, algebra must be studied for what it can be made to do. The chief aim should be the solution of concrete problems and the study of such new operations as are needed in making these solutions rather than the construction of a theoretical doctrine as an end in itself. This does not mean that theoretical doctrines should be neglected or that drill exercises in symbolic operation should be avoided. But it does mean that new operations should be introduced only as needed in the solution of concrete problems. It is only as an algebraic operation enables a pupil to solve problems that he understands as problems independent of the algebra that it is of real interest to him. Healthy interest can never be maintained for any great length of time within the *logical* boundaries of an abstract subject like algebra. How true this is will appear to each of us as we recall our early experiences. While a boy from ten to fourteen I was made to think hard about geometrical symmetry because I wanted to make a sail-boat that should sail straight. Similar triangles were of the most absorbing interest because by means of them I was enabled to measure the distance to an inaccessible point on the opposite side of a lake.

Testimony to this same general effect is at hand on all sides. May I relate just an incident. My first year of teaching was in a country school. We had a class in algebra and the pupils were drilled in the good old fashion on the fundamental operations. Four years later I visited the community and in talking with one of the pupils of that class, then grown six feet tall, we recalled old times in the school-house and he remarked that he remembered just two things from that algebra. One was the problem of finding the time between conjunctions of Mars and Jupiter, the time of revolution of each being given. The other was the problem of finding how wide a strip a farmer must cut around a given field in order to cut a certain number of acres. He confessed that he had tried lately to solve the latter problem, but found he had to look up the method. Every operation in abstract symbols, except such as anyone can do almost the first time he sees them, had vanished from his mind though he was drilled most mercilessly in them, and you may rest assured that he would never look up a single one of them for its own sake.

This started me thinking about the character of our teaching perhaps more than any other one thing. The boy had appeared interested in the fundamental operations. He could factor anything in the book and took pride in puzzling out complicated examples which I unearthed from other books. To me it appeared that he was interested in the abstract operations quite as much as in the concrete problems. But what a tremendous difference between these two interests. The factoring was a game of solitaire with a little added interest because he could excel the others and gain my approval and perhaps their envy. But interest akin to the interest in a game of solitaire is not of the kind that we should seek to develop. It is not vital. It never lasts.

The width of the strip about the field was of natural and lasting interest. The event showed clearly enough that it laid a permanent hold on the mental anatomy of the boy while the others did not. What if we were to build our whole algebra around problems of this sort? Obviously there was no thought of financial gain or any so-called practical value associated with the solution of these problems. The boy was far too shrewd to dream that he might some day make a genteel living by solving problems of this sort. It was pure, healthy intellectual interest in the problem for its own sake. It is difficult to understand how anyone could ever suspect that such problems might be thought practical in the sense that there is a dollar dangling at the end of their solution. I, for one, certainly never had the faintest trace of a belief that in their industrial activity people would be likely to be called upon to solve just such problems as these. And even if the problems were practical in this sense, that would be no great point in their favor so far as creating interest in a healthy child is concerned. Boys and girls are not economically practical. The boy plays baseball with a will and never thinks of getting paid for it. I struggled for days to get a method for measuring the distance across a lake and surely I never thought of getting rich that way.

Our fundamental proposition is then that in the beginning we will as far as possible study only such algebraic processes as we find are needed in solving concrete problems. That has been the universal rule in arithmetic and we now propose to continue it into the algebra. This proposition decides at once

the character of the whole course, the inclusion and exclusion of subject matter, the order of the topics and the general mode of treatment. An interesting verification of this is that about three years ago to my knowledge at least three different groups of people worked out an outline for elementary algebra, and while there was no intercommunication on these particular points the results were in such close agreement as to make anyone not acquainted with the facts reasonably sure that they must have come from a common source. The fact was that they arose independently of each other but from common principles.

As a curious illustration of how prone we are to find a defense for what has become endeared to us by long association, be it ever so illogical, I remark here that in some quarters objections have been raised to the new order of topics necessitated by the new point of view. The chief reason given is that the new arrangement does not naturally call for as many drill exercises in mere manipulation as did the older arrangement. This is perfectly true. The fifteen or twenty weeks of beginning algebra with practically no applications—one continuous stretch at the very outset of from 100 to 150 pages of definitions, principles and abstract exercises—these must vanish if the principles here urged are accepted. But what about the arithmetics which are used in the last four years of the grades in this respect? They contain comparatively few purely drill exercises, and an overwhelming number of concrete problems. There does not seem to be any good reason why the student in beginning algebra should need such a surfeit of abstract formal exercises while in arithmetic he has no such need. There he got an excellently working concrete problem.

To see how the dominant idea of an author determines his mode of treatment let us consider the introduction of negative numbers. If the chief purpose is to present the subject in its logical aspect he comes upon the negative number by trying to solve such equations as $x + 2 = 0$, and obtains all the properties of the negative number directly as consequences of its definition. The author whose chief purpose it is to present the subject in its dynamic aspect finds concrete instances in which the negative number is useful. He then studies its properties in connection with these concrete things to which it is applied. In this re-

spect his study of the negative number is strictly analogous to the study of integers and fractions in the grammar school. Similar differences are observable in the treatment of other topics.

This mode of treatment naturally confines itself to a somewhat smaller body of algebraic operations. But the effect is to obtain complete practical mastery of these by using them in a very large number of different connections and by awakening a genuine interest in them. When different operations are crowded upon a child as rapidly as they must be in a course where they take "everything up to quadratics" in one year there is not sufficient time to digest each one. I can think of no better illustration of this than the usual treatment of radicals. We find from thirty to forty pages of theorems, discussion and abstract exercises giving a very elaborate treatment of all thinkable cases. There is not a single concrete application. Is it your experience that when the student comes to study geometry he can handle radicals with the facility which one might desire? My experience has been that he has no idea that the complicated symbols with which he worked in that overloaded course in algebra had anything whatever to do with the altitude of an equilateral triangle.

It is proposed that we study a much smaller body of theorems on radicals—just enough to solve quadratics and then use this small body of algebraic information in solving a large body of interesting problems. We shall not know quite so much but we shall surely know something. Radicals taught this way are much more likely to become a permanent part of the learner's equipment.

The endeavor to obtain encyclopedic knowledge the first time one studies a subject but too often leads to weakness and no solid acquisitions whatever. *Seek ye first a few important things and make sure that ye find them and all other things shall be added unto you.*

While the beginning of the study of algebra should emphasize its dynamic rather than its logical aspect, it is by no means intended that it should end there. When the child has become familiar with the main processes of algebra through their dynamic aspect, they are a part of his world as much as his information about Lincoln and Christopher Columbus. They

then become legitimate objects of study for their own sake. This brings us to the necessity of a general review of the whole subject. This review must obviously be from a more mature point of view than the first course. If the two courses are essentially from the same point of view, say from the same text, then it is evident that either the first course is too difficult or the review course is not sufficiently strong to demand the student's best efforts. When we reflect a bit it is perfectly obvious that the child could not take an interest in lengthy discussions on algebraic processes the first time he studies the subject. None of us, young or old, are in the habit of reflecting deeply on things of whose very existence we are hardly aware.

Recognizing fully the great gap between the child and the mature person, it is our purpose to bridge this gap so that the transit may be normal and healthy. Beginning with the dynamic elements of algebra, which is everywhere the child's element, we learn to use its processes in doing things that the child sees some sense in doing. As the processes become familiar they themselves become objects of interest and are then studied. The great desideratum is that a natural interest is maintained all along. In this way the pupil is stimulated to greater and more pleasurable efforts. His real acquisition is much greater though he does not know so many special cases the first year. His whole intellectual life is healthier in that he is keeping in constant touch with his own common sense.

We may summarize the whole program by saying that it consists in an effort to

HUMANIZE SECONDARY MATHEMATICS.

Brown University,
Providence, R. I.